# SOLUTION OF AN INTEGER PROBLEM OF QUADRATIC PROGRAMING

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A problem of integer quadratic programing is considered. An integer linear programing problem was treated, for example, in [1 to 3]. First, we shall consider an auxiliary noninteger problem (the "continuous" problem) and then we shall show how on the basis of the known solution of the noninteger problem one can find the integer (whole number) solution. As an example, we give the problem of selecting the optimum order of external actions on a linear system.

**1. Formulation of the problem.** We are given the function

$$F(\mathbf{X}) = \mathbf{X}^* N\mathbf{X} + \mathbf{X}^*\mathbf{B} + c \tag{1.1}$$

where $\mathbf{X} = (x^1, \ldots, x^n)$ is an $n$-dimensional vector; $N$ is a real symmetric positive definite square $n$-order matrix, $B$ an $n$-dimensional vector, $c$ a real number; the asterisk * indicates transposition.

Let $n$ real numbers $\gamma_1, \ldots, \gamma_n$ be given. Out of these $n$ numbers one can construct $n!$ different $n$-dimensional vectors of which each one contains as components all the numbers $\gamma_1, \ldots, \gamma_n$. We shall denote this set of vectors by $\Omega$. One can easily think of all the points of the set $\Omega$ as lying in a plane perpendicular to the vector $(1, 1, \ldots, 1)$ and passing through the point $(a, a, \ldots, a)$, where $a = (\gamma_1 + \ldots + \gamma_n)/n$.

It is required to find a point $Z \in \Omega$, such that

$$F(\mathbf{Z}) = \min F(\mathbf{X}), \qquad \mathbf{X} \in \Omega \tag{1.2}$$

Clearly, this problem can be worked by taking all permutations, but for large $n$ this is practically impossible. Let us introduce into our consideration the set $L$ which is constructed from the set $\Omega$ in the following way: $\mathbf{X} \in L$, if

$$\mathbf{X} = \alpha_1\mathbf{X}_1 + \ldots + \alpha_s\mathbf{X}_s; \qquad \mathbf{X}_i \in \Omega, \quad \alpha_i \geqslant 0, \quad i = 1, \ldots, s; \qquad \alpha_1 + \ldots + \alpha_s = 1, \quad s \leqslant n! \tag{1.3}$$

It is obvious that $L$ is a convex closed, bounded set (a polyhedron) in an $n$-dimensional Euclidean space: $\Omega \subset L$.

First, we shall solve the auxiliary problem. We must find a point $\mathbf{Y} \in L$, such that

$$F(\mathbf{Y}) = \min F(\mathbf{X}), \qquad \mathbf{X} \in L \tag{1.4}$$

**2. Solution of the auxiliary problem.** We shall find the minimum of the function (1.1) on the set $L$. Let us denote by $\mathbf{G}(\mathbf{X})$ the gradient of the function $F(\mathbf{X})$

$$\mathbf{G}(\mathbf{X}) = \left(\frac{\partial F}{\partial x^1}, \ldots, \frac{\partial F}{\partial x^n}\right) = 2N\mathbf{X} + \mathbf{B}$$

We select an arbitrary $\mathbf{X}_1 \in L$. Evaluating the gradient of the function $F(\mathbf{X})$ at the point $\mathbf{X}_1$, we get $\mathbf{G}_1 = \mathbf{G}(\mathbf{X}_1) = 2N\mathbf{X}_1 + \mathbf{B}$. Let us find a point $\mathbf{Z}_1 \in L$, such that

$$\mathbf{Z}_1^*\mathbf{G}_1 = \min \mathbf{X}^*\mathbf{G}_1, \qquad \mathbf{X} \in L \tag{2.1}$$

The components of the vector $\mathbf{Z}_1$ have to be found in the following way. Let $\mathbf{G} = (g_1^1, \ldots, g_1^n)$, and let the largest component of the vector $\mathbf{G}_1$ be $g_1^k$. Then one must take for $z_1^k$ the smallest one of the numbers $\gamma_1, \ldots, \gamma_n$. Next, we look for the largest one of the numbers $g_1^i$ $(i \neq k)$ and for the smallest one of the remainder in the sequence $\gamma_1, \ldots, \gamma_n$, and so on, until we find all components of the vector $\mathbf{Z}_1$. It is easy to see that $\mathbf{Z}_1 \in \Omega$, i.e. the solution of the linear problem (2.1) will be an integer solution. Let us take now the linear combination

$$\mathbf{X}_1(\alpha) = \alpha\mathbf{X}_1 + (1-\alpha)\mathbf{Z}_1, \qquad \alpha \in [0, 1]$$

We find an $\alpha = \alpha_1 \in [0, 1]$, such that $F(\mathbf{X}_1(\alpha_1)) = \min F(\mathbf{X}_1(\alpha))$, $\alpha \in [0, 1]$. It is obvious that

$$\alpha_1 = \frac{-(\mathbf{X}_1 - \mathbf{Z}_1)^* [2N\mathbf{Z}_1 + \mathbf{B}]}{2(\mathbf{X}_1 - \mathbf{Z}_1)^* N (\mathbf{X}_1 - \mathbf{Z}_1)} = \frac{s_1}{s_2} \tag{2.2}$$

If $\mathbf{X}_1 \neq \mathbf{Z}_1$, then the denominator $s_2 > 0$ in (2.2) because of the positive definiteness of the matrix $N$; if, however, $\mathbf{X}_1 = \mathbf{Z}_1$ then $\mathbf{X}_1$ is a solution of the stated problem.

If the $\alpha_1$ defined by (2.2) is negative, then it is easily seen that $F(\mathbf{Z}_1) < F(\mathbf{X}_1)$. In this case, we assume that $\mathbf{X}_2 = \mathbf{Z}_1$.

Suppose that the numerator $s_1 \geqslant 0$ in (2.2). Since

$$s_2 = s_1 + (\mathbf{X}_1 - \mathbf{Z}_1)^* [2N\mathbf{X}_1 + \mathbf{B}], \qquad (\mathbf{X}_1 - \mathbf{Z}_1)^* [2N\mathbf{X}_1 + \mathbf{B}] \geqslant 0$$

(the last statement is implied by (2.1)) then $s_2 \geqslant s_1$ and $0 \leqslant \alpha_1 \leqslant 1$. In this case we set

$$\mathbf{X}_2 = \alpha_1\mathbf{X}_1 + (1-\alpha_1)\mathbf{Z}_1$$

It is obvious that $F(\mathbf{X}_2) \leqslant F(\mathbf{X}_1)$, $\mathbf{X}_2 \in L$.

We proceed in an analogous way. In this manner we obtain the sequences

$$\mathbf{X}_1, \mathbf{X}_2, \ldots, \quad \mathbf{X}_s \in L; \qquad \mathbf{Z}_1, \mathbf{Z}_2, \ldots; \quad \mathbf{Z}_s \in \Omega \tag{2.3}$$

Because of the choice of the $Z_s$ we have (2.4)

$$\varphi_s = (\mathbf{Z}_s - \mathbf{X}_s)^* \mathbf{G}_s \leqslant 0, \quad F(\mathbf{X}_1) \geqslant F(\mathbf{X}_2) \geqslant \ldots, \quad \lim F(\mathbf{X}_s) = d, \quad F(\mathbf{X}_s) \geqslant d$$

Here, $d$ is a finite number because $L$ is a bounded set.

Let $\mathbf{Y}$ be an arbitrary limit of the sequance $\mathbf{X}_1, \mathbf{X}_2, \ldots$, i.e. there exists a sequence $\mathbf{X}_{k_1}, \mathbf{X}_{k_2}, \ldots$ such that $\mathbf{X}_{k_s} \to \mathbf{Y} \in L$.

Then one can show, just as it was done in [4], that

$$F(\mathbf{Y}) = \min F(\mathbf{X}), \; \mathbf{X} \in L$$

From the results of [4] it follows that

$$\min [(\mathbf{X} - \mathbf{Y})^* \mathbf{G}] = 0, \qquad \mathbf{X} \in L, \qquad \mathbf{G} = \mathbf{G}(\mathbf{Y}) = 2N\mathbf{Y} + \mathbf{B} \tag{2.5}$$

**3. Finding an integer solution on the basis of a solution of the auxiliary problem.** We assume that a solution of the "continuous" problem is given, i.e. one knows a point $\mathbf{Y} \in L$ such that $F(\mathbf{Y}) = \min F(\mathbf{X})$, $\mathbf{X} \in L$ (Such a point can be found by the method described above).

In order to find a solution of the integer problem, i.e. to find a point $Z \in \Omega$, which yields a minimum of the function (1.1) on the set $\Omega$, we can use the following arguments:

1. Let us represent the function $F(\mathbf{X})$ in the form

$$F(\mathbf{X}) = F(\mathbf{Y}) + (\mathbf{X} - \mathbf{Y})^* [2N\mathbf{Y} + \mathbf{B}] + (\mathbf{X} - \mathbf{Y})^* N (\mathbf{X} - \mathbf{Y}) \tag{3.1}$$

Since the matrix $N$ is positive definite, we have

$$F(\mathbf{X}) \geqslant F(\mathbf{Y}) + (\mathbf{X} - \mathbf{Y})^* \mathbf{G}, \qquad \mathbf{G} = 2N\mathbf{Y} + \mathbf{B}$$

Suppose that we know the algorithm for ordering the points of the set $\Omega$ relative to the vector $\mathbf{G}$, i.e. we know a method for constructing the points

$$\mathbf{X}_{11}, \ \mathbf{X}_{12}, \ldots, \mathbf{X}_{1p_1}, \ \mathbf{X}_{21}, \ldots, \mathbf{X}_{2p_2}, \ldots \tag{3.2}$$

such that

$$\mathbf{X}_{ij} \in \Omega \qquad (i = 1, 2, \ldots; \ j = 1, 2, \ldots, p_i)$$

$$\mathbf{X}_{11}^* \mathbf{G} = \mathbf{X}_{12}^* \mathbf{G} = \ldots = \mathbf{X}_{2p_1^*} \mathbf{G} < \mathbf{X}_{21}^* \mathbf{G} = \ldots = \mathbf{X}_{2p_2^*} \mathbf{G} < \ldots \tag{3.3}$$

whereby, if $\mathbf{X}_{ij}^* \mathbf{G} = a$, then any vector $\mathbf{X} \in \Omega$, for which $\mathbf{X}^* \mathbf{G} < a$ will appear in the sequence (3.2) to the left of $\mathbf{X}_{ij}$ (it will be shown below how to construct the sequence (3.2) which has the property (3.3)). Then

$$(\mathbf{X}_{11} - \mathbf{Y})^* \mathbf{G} = \ldots = (\mathbf{X}_{1p_1} - \mathbf{Y})^* \mathbf{G} < \ldots$$

Since the point $\mathbf{Y}$ yields a minimum for the function $F(\mathbf{X})$ on the set $L$, it follows from (2.5) that $(\mathbf{X}_{ij} - \mathbf{Y})^* \mathbf{G} \geqslant 0$. We form the set

$$R_m = \{\mathbf{X}_{11}, \ldots, \mathbf{X}_{1p_1}, \ \mathbf{X}_{21}, \ldots, \mathbf{X}_{m1}, \ldots, \mathbf{X}_{mp_m}\}$$

Suppose that

$$\theta_m = \min F(\mathbf{X}), \qquad \mathbf{X} \in R_m, \qquad \delta_m = (\mathbf{X}_{m_1} - \mathbf{Y})^* \mathbf{G}$$

If $\mathbf{X} \in \Omega$ and $\mathbf{X} \overline{\in} \mathbf{R}_m$, then $(\mathbf{X} - \mathbf{Y})^* \mathbf{G} > \delta_m$. It is obvious that $\delta_m$ does not decrease, and that $\theta_m$ only decreases with an increase of $m$. For $\mathbf{X} \in \Omega$, $\mathbf{X} \overline{\in} R_m$, we have

$$F(\mathbf{X}) > F(\mathbf{Y}) + \delta_m \tag{3.4}$$

If it should happen that $\theta_m \leqslant F(\mathbf{Y}) + \delta_m$, then $\theta_m = \min F(\mathbf{X})$, $\mathbf{X} \in \Omega$.

2. The method presented above for filling the integer solution from a known "continuous" solution, is not applicable if at the point $\mathbf{Y}$ the function $F(\mathbf{X})$ attains a minimum in the entire space (then $\mathbf{G} = (0,\ldots,0)$), or if all coordinates of the vector $\mathbf{G}$ are equal. In this case, if the matrix $N$ is strictly positive definite, one can construct a sequence of points (3.2) which has the following property:

$$(\mathbf{X}_{11} - \mathbf{Y})^2 = \ldots = (\mathbf{X}_{1p_1} - \mathbf{Y})^2 < (\mathbf{X}_{21} - \mathbf{Y})^2 = \ldots = (\mathbf{X}_{2p_2} - \mathbf{Y})^2 < \ldots$$

Making use of the decomposition (3.1) and the strict positive definiteness of the matrix $N$ one can, in this case also, obtain an estimate analogous to (3.4).

4. **Example.** Let us consider the problem of selecting the optimum order of action on the linear system

$$\mathbf{X}^{\cdot}(t) = A(t)\,\mathbf{X}(t) + B(t)\,\mathbf{u}_{j(i)}(t) + \mathbf{f}(t), \qquad \mathbf{X}(0) = \mathbf{X}_0 \tag{4.1}$$

where $\mathbf{X}(t)$ is an unknown $n$-dimensional vector function; $\mathbf{A}(t)$ is an $n$-dimensional square matrix, $\mathbf{u}_{j(i)}(t)$ an $m$-dimensional vector function of the external controlled action, $\mathbf{f}(t)$ a known $n$-dimensional vector function of the external uncontrolled action, and $\mathbf{B}(t)$ an $m \times n$ matrix.

The components of the vectors $u_{j(i)}(t)$, $f(t)$ and the elements of the matrices $A(t)$ and $B(t)$ are assumed to be piece-wise continuous, given and bounded functions of time on $[0, T]$. It is sufficient that they be measurable. The vector function $\mathbf{u}_{j(i)}(t)$ is determined by the relation

$$\mathbf{u}_{j(i)}(t) = \sum_{i=1}^{q} \mathbf{K}_i \varphi(t - t_{j(i)}) \tag{4.2}$$

where $\mathbf{K}_i$ $(i = 1, \ldots, q)$ are given $m$-dimensional vectors $\varphi(t) \equiv 0$ when $t < 0$ and $t > \Delta\tau$, while when $t \in [0, \Delta r]$ the function $\varphi(t) \equiv \psi(t)$; here $\psi(t)$ is a continuous given function, $\Delta r$ is a known quantity,

$$t_k = t_{k-1} + \Delta t,\ k = 1, \ldots, q-1;\ t_0 = 0,\ t_{q-1} = T$$

where $\Delta t$ is a given quantity.

In this manner, the vector function $\mathbf{u}_{j(i)}(t)$ is determined as a discrete numerical function $j(i)$. The total number of the function $j(i)$ with $i = 1, \ldots, q$; $j = 0, \ldots, q-1$ is equal to $q!$ We shall denote the set of these functions by $\Omega_1$. We are given the functional

$$J(j(i)) = \sum_{j=0}^{q-1} \mathbf{X}^*(t_j, j(i))\ N\mathbf{X}(t_j, j(i)) \tag{4.3}$$

Here $\mathbf{X}(t_i, j(i))$ is a solution of the system (4.1) at the point $t_i$ when the control $j(i) \in \Omega_1$, is selected; the matrix $N$ is a positive definite square matrix of order $n$. It is required to find $j_0(i) \in \Omega_1$ such that

$$J(j_0(i)) = \min \sum_{j=0}^{q-1} \mathbf{X}^*(t_j, j(i))\, N\mathbf{X}(t_j, (i)), \qquad j(i) \in \Omega_1 \tag{4.4}$$

By means of Cauchy's formula the solution of the system (4.1) may be written

$$\mathbf{X}(t) = \mathbf{Y}(t)\, \mathbf{X}_0 + \int_0^t Y(t)\, Y^{-1}(\tau) \left[B(\tau) \sum_{i=1}^{q} \mathbf{K}_i \varphi(\tau - t_{j(i)}) + f(\tau)\right] d\tau$$

Here $Y(t)$ is the fundamental matrix of the homogeneous part of the system (4.1), and the problem of the minimization of the functional (4.4) can be reduced to the following one.

We are given the function

$$F(\mathbf{K}) = \mathbf{K}^* N_1 \mathbf{K} + \mathbf{K}^* \mathbf{b} + c \tag{4.5}$$

where $\mathbf{K} = (\mathbf{K}_1, \ldots, \mathbf{K}_q)$ is an $mq$-dimensional vector since $\mathbf{K}_i$ $(i = 1, \ldots, q)$ is an $m$-dimensional vector; $N_1$ is a positive definite $q$th order matrix; $\mathbf{b}$ is an $m$-dimensional vector, $c$ a real number. Besides that, we are given $qm$-dimensional vectors $\gamma_1, \ldots, \gamma_q$ from which one can construct $q!$ different $mq$-dimensional vectors which form the set $\Omega$. It is necessary to find $\mathbf{Y} \in \Omega$ such that

$$F(\mathbf{Y}) = \min F(\mathbf{K}),\ \mathbf{K} \in \Omega \tag{4.6}$$

This problem is solved by the method presented in Sections 1 and 2, but the finding of the solution of the linear problem in this case is reduced to solving the "problem of specifications". If only one component of the vectors $\gamma_i$ $(i = 1, \ldots, q)$ in (4.2) depends on the number $i$ of the reaction, then the problem coincides exactly with the one treated in Section 1.

**Appendix.** Let us consider a set $\Omega$ ordered relative to the vector $\mathbf{G}$. Below, we give an algorithm for the construction of the sequence (3.2) which has the property (3.3). For simplicity's sake, we assume that the coordinates of the vector $\mathbf{G} = (g^1, \ldots, g^s)$ are all distinct and that $g^1 < g^2 < \ldots < g^s$. By the hypothesis made in Section 1, there are no equal numbers among $\gamma_1, \ldots, \gamma_n$.

It is obvious that one must take for $\mathbf{X}_{1}$ the vector $\mathbf{X}_{1} = (x_{1}^1, \ldots, x_{1}^s)$ whose coordinates are the numbers $\gamma_i$ arranged in decreasing order.

Suppose we have constructed the beginning of the sequence (3.2): $\mathbf{X}_1, \ldots,$

$X_{k-1}$. In order to find the next vector we proceed as follows: for each of the already constructed vectors $X_1, \ldots, X_{k-1}$ we construct $(n-1)$ vectors which are obtained by a transposition of only two adjacent coordinates of this vector; from the set $D_{k-1}$ of vectors so obtained we select a vector $X_k$ such that

$$X_k{}^*G = \min X^*G, \qquad X \in D_{k-1}, \qquad X \neq X_i \qquad (i = 1, \ldots, k-1)$$

We shall show that this vector is the required one. Indeed, for each vector $Z \in \Omega$, for which

$$\min X^*G < Z^*G < \max X^*G, \quad X \in \Omega$$

there exist vectors $Z_1$ and $Z_2$ obtained from $Z$ by a transposition of only two adjacent coordinates and having the property that $Z_1{}^*G < Z^*G < Z_2{}^*G$.

But then if $X_k$ is the next vector after $X_{k-1}$ in the sequence (3.2), then there exists a vector $X_{k1}$ obtained from $X_k$ by a transposition of only two adjacent coordinates for which $X_{k1}{}^*G < X_k{}^*G$ and, since $X_k$ is the nearest vector to $X_{k-1}$ in the sequence (3.1), the vector $X_{k1}$ must occur among the vectors $X_1, \ldots X_{k-1}$. Consequently this means that the point $X_k$ is in the set $D_{k-1}$.

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